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Necessary Conditions for an Infinite Time Optimal Control Problem

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In this paper, we deal with an optimal control problem with infinite transfer time. Using methods which involve local conditional stability, we obtain necessary conditions for optimality under the assumption that the right-hand side of the state equation is Fréchet-differentiable at every point of the optimal solution, and under some weak assumptions about the asymptotical behaviour of the set of perturbations of the solution. The results are illustrated in a specific case considered by Pontryagain *et al.* © 1987 Academic Press, Inc.

INTRODUCTION

The aim of this paper is to solve some mathematical difficulties which arise when we are concerned with optimal control problems with infinite horizon. This topic has also been considered by Pontryagin, *et al.* [13], and by Halkin [9], who studied situations without given end conditions. Here, the reader will find necessary conditions for optimality which apply when the state is prescribed to approach the origin after an infinitely long evolution. They are based on results applying in the finite horizon case, previously published by Gani in [8], where he also stated conditions for optimality in infinite horizon problems. These are recalled and then improved in the present paper.

Our methods differ from those employed by the other authors in that we transform optimal control problems into problems of mathematical programming. This will be illustrated in Section 6 by the use of Pontryagin's example.

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1. Problem Statement

Let $J = [0, \infty[\subset \mathbb{R}$, E be the d-dimensional real Euclidean space, B a normed space and let there be given a set U together with two functions

$$f\colon J \times E \times B \times U \to E$$

and

$$H\colon B \to \mathbb{R}.$$

We wish to minimize the quantity $H(p)$ assuming that the absolutely continuous function $x\colon J \to E$, the function $u\colon J \to U$ and the parameter $p \in B$ verify the following conditions:

(i) $\dot{x}(t) = f(t, x(t), p, u(t))$ a.e. on J,

(ii) $x(0) = 0$,

(iii) $\lim_{t \to \infty} x(t) = 0$ and

(iv) u is an *admissible control*, i.e., for every $(x, p) \in E \times B$ the function $f_u\colon J \to E$ defined by

$$f_u(t, x, p) = f(t, x, p, u(t))$$

is locally integrable on J.

Accordingly, we shall say that the ordered triple (x_0, u_0, p_0) is an *optimal solution* if it satisfies conditions i–iv, and if the minimum of H is attained at p_0.

In this paper we shall state two sets of necessary conditions for optimality which are gathered in theorems 1 and 2, respectively. The first one has already been published in reference [8]. The second one looks very similar to it, but it introduces two important improvements:

(i) the assumptions made about the state equation are weaker, and

(ii) the transversality conditions obtained are more convenient.

2. Assumptions

We shall make five assumptions about the functions f and H.

(a) For every $(x, p, u) \in E \times B \times U$ the function f is locally integrable on J.

(b) There exist functions $\theta \in L^1_{\mathrm{Loc}}(J)$ and $k \in L^1(J)$ satisfying

$$\|f(t, x, p, u) - f(t, y, q, u)\| \leqslant \theta(t)\|x - y\| + k(t)\|p - q\|$$

for all $(t, x, y, p, q, u) \in J \times E^2 \times B^2 \times U$.

(c) For every $t \in J$ we have

(i) the function f_{u_0} is Fréchet-differentiable at $(x_0(t), p_0)$, and

(ii) the function $A: J \to \mathscr{L}(E, E)$ defined by

$$A(t) = \partial_x f(t, x_0(t), p_0, u_0(t))$$

satisfies

$$\|f(t, x, p, u_0(t)) - f(t, y, p, u_0(t)) - A(t)(x - y)\| \leqslant k(t)\|x - y\|$$

for all $(x, y, p) \in E^2 \times B$.

(d) Let $r: J^2 \to \mathscr{L}(E, E)$ be the absolutely continuous function defined by

$$r(t, t) = I \quad \text{(the identity)} \qquad \text{for all } t \in J, \text{ and}$$

$$\frac{dr}{dt}(t, s) = A(t)\, r(t, s) \qquad \text{a.e. on } J,\ s \in J.$$

We assume that there exist $c > 0$ and two supplementary subspaces V and W of E such that we have

(i) $\lim_{t \to \infty} r(t, s)\, x = 0$ for all $x \in V$ and $s \in J$,

(ii) $\|r(t, s)\, x\| \leqslant c\|x\|$ for all $x \in V$ and $t \geqslant s$, and

(iii) $\|r(t, s)\, x\| \leqslant c\|x\|$ for all $x \in W$ and $s \geqslant t$.

We shall denote by P_v and P_w the canonical projection of E on V and W, respectively.

Remark. We can choose V to be of maximal dimension.

(e) The function H satiesfies the following condition: there exists a sublinear function $H^0: B \to \mathbb{R}$ such that for every d-simplex π contained in B there exists a positive h_π such that

$$\lim_{h \downarrow 0} H^h = H^0 \qquad \text{uniformly on } \pi,$$

where

$$H^h(p) = (1/h)(H(p_0 + hp) - H(p_0)),$$

and H^h is continuous when restricted to π, $h \in]0, h_\pi]$.

Remark. Assumption (e) is a hypothesis on the Gâteau-derivative of H. For instance, if H is convex or if H has a Fréchet-derivative, then this assumption is verified.

3. A First Maximum Principle

Let $\mathscr{D}'$ be the space of distributions over $\mathbb{R}$. Furthermore, let N be the subset of J where

$$\dot{x}_0(t)=f(t, x_0(t), p_0, u_0(t)).$$

(Clearly, $J\backslash N$ and hence N are measurable). Finally, let $\Omega \subset N \times U$ be defined by

$$\Omega=\left\{(t, u); f(t, x_0(t), p_0, u)=\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} f(\tau, x_0(t), p_0, u)\, d\tau\right\}.$$

Then we have the following theorem.

Theorem 1. *If* (x_0, u_0, p_0) *is an optimal solution satisfying assumptions* (a)–(e), *then there exist* $T \in J$, $a \in \mathbb{R}$ *and a function* $\psi\colon \mathbb{R} \to E$, $(a, \psi) \neq (0, 0)$, *such that we have*:

(1) $a \geqslant 0$,

(2) ψ *is absolutely continuous and the support of* $\dot{\psi}$ *is in* J,

(3) *in* $\mathscr{D}'$ *we have*

$$\dot{\psi}=-A^* \psi$$

where the function $A\colon R \to \mathscr{L}(E, E)$ *is defined by*

$$A(t)=\begin{cases}\partial_x f(t, x_0(t), p_0, u_0(t)) & \text{a.e. on } J \\ 0 & \text{on }]-\infty, 0[\end{cases},$$

(4) *the function* $\mathscr{M}\colon J \times U \to \mathbb{R}$ *defined by*

$$\mathscr{M}(t, u)=\psi(t) \cdot f(t, x_0(t), p_0, u)$$

satisfies

$$\mathscr{M}(t, u) \geqslant \mathscr{M}(t, u_0(t)) \qquad \text{for all } (t, u) \in \Omega.$$

(5) $\psi(T) \cdot V=0$.

(6) *The function* $K\colon B \to \mathbb{R}$ *defined by*

$$K(p)=aH(p)+\int_0^T \psi(s) \cdot f(s, x_0(s), p, u_0(s))\, ds$$

$$+\int_T^{\infty} \psi(s) \cdot P_w\{f(s, x_0(s), p, u_0(s))-f(s, x_0(s), p_0, u_0(s))\}\, ds$$

is Gâteau-differentiable at p_0 *and we have*

$$\partial_p K(p_0) \cdot p \geqslant 0 \qquad \text{for all } p \in B.$$

4. The Improved Case

In the following, we shall state necessary conditions under somewhat weaker assumptions. We shall not require that the derivative $\partial_x f$ has the restrictive properties (c) and (d), but only that there exists a function with these properties. More precisely, instead of (c) and (d), we make the following assumptions (c′) and (d′).

(c′) For every $t \in J$ we have

(i) the function f_{u_0} is F-differentiable at $(x_0(t), p_0)$ and

(ii) there exists a function $\mathscr{A}: J \to \mathscr{L}(E, E)$ verifying

$$\|f(t, x, p, u_0(t)) - f(t, y, p, u_0(t)) - \mathscr{A}(t)(x-y)\| \leqslant k(t)\|x-y\| \qquad \text{for all } (x, y, p) \in E^2 \times B.$$

(d′) Let $r: J^2 \to \mathscr{L}(E, E)$ be the absolutely continuous function defined by

$$r(t, t) = I \qquad \text{for all } t \in J$$

$$\text{and} \quad \frac{dr}{dt}(t, s) = \mathscr{A}(t)\, r(t, s) \qquad \text{a.e. on } J,\ s \in J.$$

We assume that there exist $c > 0$ and two supplementary subspaces V and W of E such that we have

(i) $\lim_{t \to \infty} r(t, s)\, x = 0$ for all $x \in V$ and $s \in J$

(ii) $\|r(t, s)\, x\| \leqslant c\|x\|$ for all $x \in V$ and $t \geqslant s$ and

(iii) $\|r(t, s)\, x\| \leqslant c\|x\|$ for all $x \in W$ and $s \geqslant t$.

The main advantage of this new version is, that $\mathscr{A}$ admits $\mathscr{L}_1$-perturbations, whereas, in the old version, A was a fixed functional.

Theorem 2. *Let (x_0, u_0, p_0) be an optimal solution satisfying conditions (a), (b), (c′), (d′) and (e). Then there exist $a \in \mathbb{R}$, a function $\psi: \mathbb{R} \to E$, $(a, \psi) \neq (0, 0)$, and two linear functions $A_0 \in \mathscr{L}(V, E)$ and $\mathscr{E}_0 \in \mathscr{L}(B, E)$ such that we have:*

(1) $a \geqslant 0$,

(2) *ψ is absolutely continuous and the support of $\dot{\psi}$ is in J,*

(3) *in $\mathscr{D}'$ we have*

$$\dot{\psi} = -A^*\psi$$

where $A: \mathbb{R} \to \mathscr{L}(E, E)$ *is defined by*

$$A(t) = \begin{cases} \partial_x f(t, x_0(t), p_0, u_0(t)) & \text{a.e. on } J \\ 0 & \text{on }]-\infty, 0[. \end{cases}$$

(4) *The function* $\mathscr{M}: J \times U \to \mathbb{R}$ *defined by*

$$\mathscr{M}(t, u) = \psi(t) \cdot f(t, x_0(t), p_0, u)$$

satisfies

$$\mathscr{M}(t, u) = \mathscr{M}(t, u_0(t)) \qquad \text{for all } (t, u) \in \Omega.$$

(5) $\Lambda_0 V$ *and* V *have the same dimension and we have*

$$\psi(0) \cdot \Lambda_0 V = 0.$$

(6) *The function* $K: B \to \mathbb{R}$ *defined by*

$$K(p) = a\, \partial_p H(p_0) \cdot p - \psi(0)\, \mathscr{E}_0(p)$$

verifies

$$K(p) \geqslant 0 \qquad \text{for all } p \in B.$$

(7) *For every* $(\alpha, p) \in V \times B$ *and for large* T *there exists a unique bounded and continuous solution*

$$\xi_{\alpha, p, T}: J \to E$$

of the following integral equation:

$$z(t) = r(t, T)\, \alpha + \int_T^t r(t, s)\, P_v \{(A(s) - \mathscr{A}(s))\, z(s) + b(s) \cdot p\}\, ds$$

$$- \int_t^\infty r(t, s)\, P_w \{(A(s) - \mathscr{A}(s))\, z(s) + b(s) \cdot p\}\, ds,$$

where the function $b: J \to \mathscr{L}(B, E)$ *is defined by*

$$b(t) = \partial_p f(t, s_0(t), p_0, u_0(t)), \qquad t \in J,$$

and

(8) *the operators* Λ_0 *and* $\mathscr{E}_0$ *satisfy*

$$\Lambda_0(\alpha) = \xi_{\alpha, 0, T}(0) \qquad \text{for all } x \in V$$

and

$$\mathscr{E}_0(p) = \xi_{0,p,T}(0) \qquad \text{for all } p \in B,$$

respectively.

5. Proof of Theorem 2

To prove the theorem we shall proceed in four steps. We first show the existence of fixed points for a compact operator working on the unit sphere of the space of continuous functions satisfying the terminal condition. This permits us to construct a family of absolutely continuous solutions of the differential equation in step two. Next, we establish some regularity conditions of this family which enable us to transform the initial problem into one with finite horizon. Finally, we obtain the stated necessary conditions for optimality by applying a previously established maximum principle to the modified problem.

5.1. *Fixed Points*

Let $r\colon J^2 \to \mathscr{L}(E, E)$ be the absolutely continuous function defined by

$$r(t, t) = I \qquad \text{for all } t \in J$$

and

$$\frac{dr}{dt}(t, s) = \mathscr{A}(t)\, r(t, s) \qquad \text{a.e. on } J,\ s \in J.$$

In view of assumption (d′) there exist two complementary subspaces V and W of E and two constants $c > 0$ and $c_1 \geq 0$ such that we have

$$\|r(t, s)\, x\| \leq c\|x\| \qquad \text{for all } x \in V \text{ and all } t \geq s,$$

$$\|r(t, s)\, x\| \leq c\|x\| \qquad \text{for all } x \in W \text{ and all } s \geq t,$$

$$\|r(t, s)\, P_V\| \leq c_1 \qquad \text{for all } t \geq s$$

and

$$\|r(t, s)\, P_W\| \leq c_1 \qquad \text{for all } s \geq t.$$

From now on we shall assume that $T \in J$ is large enough, i.e., that we have

$$c_1 \int_T^\infty k(s)\, ds \leq \frac{1}{3}.$$

Next, let σ be the unit sphere of the normed space B, $S \subset E$ the sphere defined by

$$S = \{\alpha \in V,\ c\|\alpha\| \leqslant 1/3\}$$

and $\mathscr{C}_0$ the space of all continuous functions $z: J \to E$ satisfying $\lim_{t \to \infty} z(t) = 0$, normed by

$$\|z\| = \sup_{t \in J} \|z(t)\|.$$

Finally, let us define functions $K_{\alpha,p,T}: \mathscr{C}_0 \to \mathscr{C}_0$, $\alpha \in S$, $p \in \sigma$, $T > 0$ by:

$$K_{\alpha,p,T}(z)_{(t)} = r(t, T)\,\alpha + \int_T^t r(t, s)\, g_V(s, z(s), p)\, ds - \int_t^\infty r(t, s)\, g_W(s, z(s), p)\, ds$$

for $t \geqslant T$ and

$$= \alpha - \int_T^\infty r(t, s)\, g_W(s, z(s), p)\, ds$$

for $t < T$, where

$$g_V(t, z, p) = P_v\{f(t, x_0(t) + z, p_0 + p, u_0(t)) \\ - f(t, x_0(t), p_0, u_0(t)) - \mathscr{A}(t)\, z\}$$

and

$$g_W(t, z, p) = P_w\{f(t, x_0(t) + z, p_0 + p, u_0(t)) \\ - f(t, x_0(t), p_0, u_0(t)) - \mathscr{A}(t)\, z\},$$

respectively. Then we have

$$\|K_{\alpha,p,T}(z)\| \leqslant c\|\alpha\| + c_1(\|z\| + \|p\|) \int_T^\infty k(s)\, ds \qquad \text{on } S \times \sigma$$

and, in view of Lebesgue's theorem,

$$\lim_{t \to \infty} \|K_{\alpha,p,T}(z)_{(t)}\| = 0 \qquad \text{uniformly on every sphere of } \mathscr{C}_0.$$

Moreover, the following lemma is true.

LEMMA. *For every $\alpha \in S$ and $p \in \sigma$ the function $K_{\alpha,p,T}$ is compact, transforms the unit sphere of $\mathscr{C}_0$ into itself and hence, has a fixed point $z_{\alpha,p,T} \in \mathscr{C}_0$.*

Proof. Indeed we have

(i) $K_{\alpha,p,T}$ is clearly absolutely continuous on J,

(ii) $$\begin{aligned}\frac{d}{dt} K_{\alpha,p,T}(z)_{(t)} &= \mathscr{A}(t)\, K_{\alpha,p,T}(z)_{(t)} + f(t, x_0(t) + z(t), p_0 + p, u_0(t)) \\ &\quad - f(t, x_0(t), p_0, u_0(t)) - \mathscr{A}(t) z(t) \quad \text{a.e. on } [T, \infty[\\ &= 0 \quad \text{a.e. on } [0, T[,\end{aligned}$$

(iii) $$\left\| \frac{d}{dt} K_{\alpha,p,T}(z)_{(t)} \right\| \leqslant \|\mathscr{A}(t)\| + 2k(t).$$

Then, the compactness of $K_{\alpha,p,T}$ is a consequence of Ascoli's theorem.

5.2. *A family of Solutions*

In the following we shall construct a family $\{x_{\alpha,p,T};\ \alpha \in S, p \in \sigma\}$ of solutions satisfying

$$\dot{x}(t) = f(t, x(t), p_0 + p, u_0(t)) \qquad \text{a.e. on } J$$

and

$$\lim_{t \to \infty} x(t) = 0.$$

In view of the preceding lemma we have for all $\alpha \in S$ and $p \in \sigma$: $z_{\alpha,p,T}$ is absolutely continuous on J and

$$\begin{aligned}\dot{z}_{\alpha,p,T} &= f(t, x_0(t) + z_{\alpha,p,T}(t), p_0 + p, u_0(t)) - f(t, x_0(t), p_0, u_0(t)) \\ &\qquad \text{a.e. on } [T, \infty[\\ &= 0 \qquad \text{a.e. on } [0, T].\end{aligned}$$

Therefore, the absolutely continuous function $x_{\alpha,p,T}: J \to E$, defined by

$$x_{\alpha,p,T}(T) = x_0(T) + z_{\alpha,p,T}(T)$$

and

$$\dot{x}_{\alpha,p,T}(t) = f(t, x_{\alpha,p,T}(t), p_0 + p, u_0(t)) \qquad \text{a.e. on } J$$

is equal to

$$x_{\alpha,p,T}(t) = x_0(t) + z_{\alpha,p,T}(t) \qquad \text{on } [T, \infty[.$$

Hence, we have

$$\lim_{t \to \infty} x_{\alpha,p,T}(t) = 0,$$

which completes the construction of $\{x_{\alpha,p,T}\}$.

5.3. *Regularity Conditions*

By definition of $K_{\alpha,p,T}$ and the properties of r and (c′) there exists $M>0$ such that

$$\|x_{\alpha,p,T}-x_{\beta,q,T}\| \leqslant M(\|\alpha-\beta\|+\|p-q\|) \qquad \text{for all} \quad (\alpha,\beta,p,q)\in S^2\times\sigma^2$$

Hence, for every $\alpha\in S$ and $p\in\sigma$ the solution $x_{\alpha,p,T}$ is unique and satisfies obviously the following condition:

$$\|x_{\alpha,p,T}-x_0\| \leqslant M(\|\alpha\|+\|p\|).$$

So we have:

$$\|z_{\alpha,p,T}\| \leqslant M\cdot(\|\alpha\|+\|p\|).$$

In the following we shall linearize the problem in order to compute the derivative of $x_{\alpha,p,T}$ at the origin of $S\times\sigma$. Let us therefore define the function $\mathscr{L}_{\alpha,p,T}\colon \mathscr{C}_0\to\mathscr{C}_0$ by

$$\mathscr{L}_{\alpha,p,T}(z)_{(t)} = r(t,T)\,\alpha+\int_T^t r(t,s)\,P_v\{(A(s)-\mathscr{A}(s))\,z(s)+b(s)\cdot p\}\,ds$$

$$-\int_t^\infty r(T,s)\,P_w\{(A(s)-\mathscr{A}(s))\,z(s)+b(s)\cdot p\}\,ds$$

$$\text{for all } \alpha\in S,\ p\in\sigma \text{ and } t\geqslant T$$

$$=\alpha-\int_T^\infty r(T,s)\,P_w\{(A(s)-\mathscr{A}(s))\,z(s)+b(s)\cdot p\}\,ds$$

$$\text{for all } \alpha\in S,\ p\in\sigma \text{ and } t<T.$$

For every fixed $\alpha\in S$ and $p\in\sigma$ this function transforms the sphere $\{\|z\|\leqslant 1\}$ of $\mathscr{C}_0$ into itself and is compact. Hence, $\mathscr{L}_{\alpha,p,T}$ has a fixed point $\zeta_{\alpha,p,T}$, i.e.,

$$\zeta_{\alpha,p,T}(t)=\mathscr{L}_{\alpha,p,T}(\zeta_{\alpha,p,T})(t) \qquad \text{for all } t\in[0,\infty[.$$

So $\zeta_{\alpha,p,T}$ is absolutely continuous on $[0,\infty[$ and we have:

$$\frac{d}{dt}\zeta_{\alpha,p,T}(t)=A(t)\,\zeta_{\alpha,p,T}(t)+b(t)\cdot p \qquad \text{a.e. on } [T,\infty[$$

$$=0 \qquad \text{on } [0,T[.$$

From the existence of $M'>0$, satisfying the following inequality

$$\|\zeta_{\alpha,p,T}-\zeta_{\beta,q,T}\| \leqslant M'\{\|\alpha-\beta\|+\|p-q\|\} \qquad \text{for } \{\alpha,\beta,p,q\}\in S^2\times\sigma^2,$$

we deduce the uniqueness of the solution $\zeta_{\alpha,p,T}$, $\alpha \in S$, $p \in \sigma$. So we have obviously

$$\|\zeta_{\alpha,p,T}\| \leqslant M'\{\|\alpha\| + \|p\|\},$$

and $\zeta_{\alpha,p,T}$ is linear and continuous in $\{\alpha, p\}$ on $S \times \sigma$. Now, let $\xi_{\alpha,p,T}$: $[0, \infty[\to E$, $\alpha \in S$, $p \in \sigma$ be defined by

(i) $\xi_{\alpha,p,T}$ is absolutely continuous on $[0, \infty[$,

(ii) $\xi_{\alpha,p,T}(T) = \zeta_{\alpha,p,T}(T)$,

(iii) $\dfrac{d}{dt}\xi_{\alpha,p,T}(t) = A(t)\,\xi_{\alpha,p,T}(t) + b(t)\cdot p \qquad$ a.e. on $[0, \infty[$

and

(iv) $\xi_{\alpha,p,T}$ is linear in $\{\alpha, p\}$ on $V \times B$.

Then, we have

$$\xi_{\alpha,p,T}(t) = \zeta_{\alpha,p,T}(t) \qquad \text{for all } t \in [T, \infty[, \alpha \in S \text{ and } p \in \sigma,$$

and, hence:

$$\lim_{t \to \infty} \xi_{\alpha,p,T}(t) = 0 \qquad \text{for all } \alpha \in V \text{ and } p \in B.$$

We deduce the uniqueness of the solution $\xi_{\alpha,p,T}$, $\alpha \in V$, $p \in B$.

Moreover, $\xi_{\alpha,p,T}$ is obviously the unique bounded and continuous solution for the following integral equation

$$z(t) = r(t, T)\,\alpha + \int_T^t r(t, s)\,P_v\{(A(s) - \mathscr{A}(s))\,z(s) + b(s)\cdot p)\}\,ds$$

$$- \int_t^\infty r(t, s)\,P_w\{(A(s) - \mathscr{A}(s))\,z(s) + b(s)\cdot p\}\,ds \text{ on } [0, \infty[$$

$\alpha \in V$, $p \in B$.

We have by Lebesgue's theorem

$$x_{\alpha,p,T}(t) = x_0(t) + \xi_{\alpha,p,T}(t) + o(\|\alpha\| + \|p\|) \qquad \text{on } [0, \infty[\times S \times \sigma$$

and uniformly in t.

Consequently, the following three functions

$$\Xi_T\colon [0, \infty[\times S \times \sigma \to E,\ \Lambda_T \in \mathscr{L}(V, E) \text{ and } \mathscr{E}_T \in \mathscr{L}(B, E)$$

are well defined by

$$\Xi_T(t, \alpha, p) = x_{\alpha,p,T}(t),$$
$$\Lambda_T(\alpha) = \xi_{\alpha,0,T}(T)$$

and

$$\mathscr{E}_T(p) = \xi_{0,p,T}(T),$$

respectively. They satisfy the identity

$$\Xi_T(t, \alpha, p) = x_0(t) + R(t, T)\{\Lambda_T(\alpha) + \mathscr{E}_T(p)\} + \int_T^t R(t, s)\{b(s) \cdot p\}\, ds$$
$$+ o(\|\alpha\| + \|p\|)$$

on $[0, \infty[\times S \times \sigma$ and uniformly in t, where

$$R\colon J^2 \to \mathscr{L}(E, E)$$

is the absolutely continuous function defined by

$$R(t, t) = I$$

and

$$\frac{d}{dt} R(t, s) = A(t)\, R(t, s).$$

We have obviously

$$\|\Lambda_T(\alpha) - \alpha\| + \|\mathscr{E}_T(p)\| \leqslant 3c \int_T^\infty k(s)\, ds \qquad \text{on } S \times \sigma$$

and, hence,

(i) $\lim\limits_{T \to \infty} \Lambda_T P_v = P_v$

and

(ii) $\lim\limits_{T \to \infty} \mathscr{E}_T = 0.$

Therefore, for T large enough the function Λ_T conserves the dimension of V.

5.4. *The modified Problem*

In view of the preceding paragraphs we now fix $\tau \geqslant T$ and, restricting our attention to the interval $J_\tau = [0, \tau]$, consider the following modified problem of optimization: minimize the quantity $H(p_0 + p)$ assuming that the absolutely continuous function $x: J_\tau \to E$, the function $u: J_\tau \to U$ and the parameters $\alpha \in S$, $p \in \sigma$ verify the following conditions

(i) $\dot{x}(t) = f(t, x(t), p_0 + p, u(t))$,

(ii) $x(0) = 0$

(iii) $x(\tau) = \Xi_T(\tau, \alpha, p)$

and

(iv) for every $(x, p) \in E \times \sigma$ the function $f_u: J_\tau \times E \times \sigma$ defined by

$$f_u(t, x, p) = f(t, x, p_0 + p, u(t))$$

is integrable on J_τ.

From the optimality of (x_0, u_0, p_0) we conclude that the minimum is attained when

(i) $x = x_0|_{J_\tau}$

(ii) $u = u_0|_{J_\tau}$

(iii) $\alpha = 0$

and

(iv) $p = 0$.

Thus (x_0, u_0) when restricted to J_τ, is an optimal solution for the modified problem and hence satisfies all necessary conditions, especially those of [8, Theorem 2] namely:

There exist a real a_τ and an absolutely continuous function $\psi_\tau: \mathbb{R} \to E$, $(a_\tau, \psi_\tau) \neq (0, 0)$, such that

(1′) $a_\tau \geqslant 0$,

(2′) the support of $\dot{\psi}_\tau$ is in J_τ,

(3′) $\dot{\psi}_\tau = -A^*\psi_\tau$ in $\mathscr{D}'$, where $A_\tau: \mathbb{R} \to \mathscr{L}(E, E)$ is defined by

$$\begin{aligned} A_\tau(t) &= \partial_x f(t, x_0(t), p_0, u_0(t)) && \text{a.e. on } J_\tau \\ &= 0 && \text{on } \mathbb{R}\backslash J_\tau, \end{aligned}$$

(4′) the function $\mathscr{H}_\tau: J_\tau \times E^2 \times B \times U \times \mathbb{R} \to \mathbb{R}$ defined by $\mathscr{H}_\tau(t, x, \psi, p, u, a) = aH(p_0 + p) + \psi \cdot f(t, x, p_0 + p, u)$ and the function $\mathscr{M}_\tau: J_\tau \times U \to R$ defined by $\mathscr{M}_\tau(t, u) = \mathscr{H}_\tau(t, x_0(t), \psi_\tau(t), 0, u, a_\tau)$ satisfies

$$\mathscr{M}_\tau(t, u) \geqslant \mathscr{M}_\tau(t, u_0(t)) \qquad \text{for all } (t, u) \in (\Omega \cap J_\tau) \times U,$$

(5′) the functions $K_\tau : S \times \sigma \to \mathbb{R}$ defined by

$$K_\tau(\alpha, p) = \int_0^\tau \mathscr{H}_\tau(t, x_0(t), \psi_\tau(t), p, u_0(t), a_\tau)\, dt - \psi_t(\tau)\, \Xi_\tau(\tau, \alpha, p)$$

are Gâteaux differentiable at (0, 0), and the quantity

$$\partial_\alpha K_\tau(0, 0) \cdot \alpha + \partial_p K_\tau(0, 0) \cdot p$$

is nonnegative for all $(\alpha, p) \in V \times B$. As a direct consequence of 5′ we have

(6′) $\psi_\tau(\tau) \cdot R(\tau, T)\, \Lambda_T V = 0$ and

$$\int_0^\tau (a_\tau \partial_p H(p_0) \cdot p + \psi_\tau(s)\{b(s) \cdot p\})\, ds$$

$$-\psi_\tau(\tau) \left\{ R(\tau, T)\, \mathscr{E}_T(p) + \int_T^\tau R(\tau, s)\{b(s) \cdot p\}\, ds \right\} \geqslant 0, \qquad p \in B.$$

Finally, we transfer the terminal condition to the origin in statement 7′.

(7′) Defining functions $\Lambda_0 \in \mathscr{L}(V, E)$ and $\mathscr{E}_0 \in \mathscr{L}(B, E)$ by:

$$\Lambda_0 = R(0, T)\, \Lambda_T(p),$$

$$\mathscr{E}_0(p) = R(0, T)\, \mathscr{E}_R(p) - \int_0^T R(0, s)\{b(s) \cdot p\}\, ds, \qquad p \in B,$$

respectively, we have

(i) $\Lambda_0(\alpha) = \xi_{\alpha, 0, T}(0), \quad \alpha \in V,$

(ii) $\mathscr{E}_0(p) = \xi_{0, p, T}(0) \quad p \in B,$

(iii) $\psi_\tau(0) \cdot \Lambda_0 V = 0$ and

(iv) $\tau\, a_\tau \partial_p H(p_0) \cdot p - \psi_\tau(0)\, \mathscr{E}_0(p) \geqslant 0, \quad p \in B.$

To end the proof let (a, ψ) be a limit point of the sequence $\{n\, a_n, \psi_n\}$, $n \in \mathbb{N}$, normalized by:

$$na_n + \|\psi_n(0)\| = 1.$$

Then it is easy to see that (a, ψ) satisfies all the conditions stated in the theorem.

6. Pontryagin's Example

The following problem stems from reference [13, Chap. 4, Sect. 24], where it has been stated but not solved.

Let there be given an initial state x_0, a terminal state x_1 and the following state equation:

$$\frac{dx}{dt}(t) = f(t, x(t), u(t)) \text{ a.e. on } \mathbb{R}_+,\ u(t) \in U,$$

where U is a given domain of values. The aim is to give necessary conditions for an admissible control function $u^*(t)$ to generate an optimal solution $x^*(t)$ of the state equation, i.e., $(x^*(t), u^*(t))$ minimizes the performance index

$$p = \int_0^\infty f^0(t, x(t), u(t))\, dt$$

among all those $(x(t), u(t))$, which satisfy the following four conditions:

(i) $x(t)$ and $u(t)$ are defined and $x(t)$ is absolutely continuous on $\mathbb{R}_+$,

(ii) $x(0) = x_0$,

(iii) $\lim_{t \to \infty} x(t) = x_1$,

(iv) the integral p is defined.

Clearly, this problem can not be solved by straightforward application of our method. Consequently, we shall adapt it to the previously prescribed frame.

Therefore, take any $h \in L^1(\mathbb{R}_+, \mathbb{R})$, such that $\int_{\mathbb{R}_+} h(s)\, ds = 1$. For every $u(t)$ and the corresponding trajectory $x(t)$ satisfying conditions (i)–(iv) define function Z by

$$Z(t) = (z^0(t), z(t)), \qquad t \in \mathbb{R}_+$$

where

$$z^0(t) = \int_0^t f^0(s, x(s), u(s))\, ds - p \int_0^t h(s)\, ds$$

and

$$z(t) = x(t) - \left(1 - \int_0^t h(s)\, ds\right) x_0 - \left(\int_0^t h(s)\, ds\right) x_1.$$

We see immediately, that Z is absolutely continuous on $\mathbb{R}_+$ and satisfies

$$\frac{dz^0}{dt}(t) = f^0(t, z(t) + (1 - b(t))\, x_0 + b(t)\, x_1, u(t)) - p \cdot h(t)$$

and

$$\frac{dz}{dt}(t) = f(t, z(t) + (1 - b(t))\, x_0 + b(t)\, x_1, u(t)) - h(t) \cdot (x_1 - x_0),$$

where $b(t)$ is defined by

$$b(t) = \int_0^t h(s)\, ds.$$

Now, if we define function G by

$$G = (g^0, g),$$

where $g^0 = g^0(t, Z, p, u)$ and $g = g(t, Z, p, u)$ are equal to the right-hand sides of the preceding two equations, respectively, then Pontryagin's problem takes the following form.

Minimize p s.t.

Z is absolutely continuous on $\mathbb{R}_+$,

$$\frac{dZ}{dt}(t) = G(t, Z(t), p, u(t)) \text{ a.e.,}$$

$$Z(o) = 0,$$

$$\lim_{t \to \infty} Z(t) = 0,$$

where 0 is the origin of the augmented state space.

But this is our initial problem, stated in Section 1 of the present paper and, hence, Theorem 2 applies!

Note that Pontryagin's problem leads to the particularly simple situation, where the objective function H is the identity.

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